

# A CONDITION ENSURING SPATIAL CURVES DEVELOP TYPE-II SINGULARITIES UNDER CURVE SHORTENING FLOW

GABRIEL J. H. KHAN

ABSTRACT. We show that if a curve that any curve immersed in  $\mathbb{R}^3$  has everywhere positive torsion after any inflection points, it develops a type-II singularity under curve shortening flow.

## 1. INTRODUCTION

Let  $\gamma$  be a smooth immersion from  $S^1$  to  $R^n$ . We then define the following differential equation.

$$(1) \quad \partial_t \gamma = \kappa N$$

where  $\kappa$  is the curvature and  $N$  is the unit normal vector. We will study solutions to this equation, which consist of a family of curves  $\gamma_t$  with  $t \in [0, \omega)$  which satisfy (1) with  $\gamma_0 = \gamma$ .

This is commonly referred to as curve shortening flow and is the simplest example of mean curvature flow. Michael Gage and Richard Hamilton proved short time existence and analyticity of the flow [4] and Matthew Grayson proved that the flow continues so long as curvature remains bounded [5]. However, since the flow is the  $L^2$  gradient flow for length of the curve (hence the name), a singularity must emerge as some time  $\omega$ . A blow-up singularity is Type I if  $\lim_{t \rightarrow \omega} M_t \cdot (\omega - t)$  is bounded and Type II otherwise where

$$(2) \quad M_t = \sup_{p \in \gamma_t} \kappa^2(p).$$

We use the results of [2] throughout and assume familiarity with this work.

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## 2. THEOREM AND PROOF

**Theorem 1.** *Given a curve  $\gamma$  in  $\mathbb{R}^3$  which not contained in any plane, under the curve shortening flow  $\gamma_t$  develops a type II singularity.*

*Proof.* Suppose  $\gamma_0$  does not lie in any plane. Then, for all  $t$  where the flow is defined, the torsion  $\tau$  is non-zero as the flow is real-analytic so without loss of generality, the torsion is not everywhere zero.

Now suppose  $\gamma$  develops a type I singularity at time  $\omega$ . Then, under renormalization as described in [2],  $\gamma_t$  approaches an Abresch-Langer solution [1] with finite winding number in the  $C^\infty$  sense [2]. For any such Abresch-Langer curve  $S$ ,  $\sup_{p \in S} \kappa(p) \cdot L < \infty$  where  $L$  is the length of  $S$ , i.e.

$$L = \int_S ds.$$

Since  $\gamma$  converges to some  $S$ , the functional  $D(t) = \sup \kappa_t \cdot L_t$  for  $t \in [0, \omega)$  and converges to a finite limit at  $\omega$ . Let  $D = \lim_{t \rightarrow \omega} D(t)$ .

Furthermore, given that  $\gamma$  develops a type-I singularity, all blow up sequences are essential and since the curve converges to an Abresch Langer solution, any sequence  $p_m, t_m$  such that  $\lim_{m \rightarrow \infty} t_m = \omega$ , by the rough planarity theorem of [2],

$$\limsup_{t \rightarrow \omega} \sup_{p \in \gamma_t} \frac{\tau}{\kappa}(p) = 0$$

where  $\tau$  is the torsion of  $\gamma_t(p)$ .

A consequence of the fact that all sequences are essential is that there exists a time  $c \in [0, \omega)$  such that for all  $t \in [c, \omega)$ ,  $\gamma_t$  has no inflection points so torsion is defined everywhere on the curve. Suppose torsion is positive for all time after  $c$ .

Now we consider the  $L^1$  norm of torsion on the curve  $\gamma_t$  for  $t \in [c, \omega)$ .

$$\|\tau\|_1 = \int_{\gamma_t} |\tau| ds$$

We parametrize  $\gamma_t$  smoothly by  $u \in [0, 2\pi)$  and utilize the calculations from [2] and [3] for the following few calculations.

$$\int_{\gamma_t} |\tau| ds = \int_0^{2\pi} |\tau| \cdot v du \text{ where } v^2 = \langle \partial_u \gamma_t, \partial_u \gamma_t \rangle.$$

Taking the derivative with respect to time, we obtain

$$\partial_t \int_{\gamma_t} |\tau| ds = \partial_t \int_0^{2\pi} (|\tau| \cdot v) du = \int_0^{2\pi} \partial_t (|\tau| \cdot v) du$$

We then calculate this explicitly.

$$\begin{aligned} \int_{\gamma} \partial_t |\tau| \cdot v du &= \int_{\gamma} \partial_t (\tau \cdot v) du \\ &= \int_0^{2\pi} (\partial_t \tau) \cdot v + (\partial_t v) \cdot \tau du \\ &= \int_0^{2\pi} \left( 2\kappa^2 \tau + \partial_s \left( \frac{2\tau}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau \right) \cdot v \\ &\quad + -\kappa^2 v \cdot \tau du \\ &= \int_0^{2\pi} \kappa^2 \tau v du + \int_0^{2\pi} \left( \partial_s \left( \frac{2\tau}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau \right) \cdot v du \\ &= \int_{\gamma} \kappa^2 \tau ds + \int_{\gamma} \partial_s \left( \frac{2\tau}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau ds \\ &= \int_{\gamma} \kappa^2 \tau ds + \left( \frac{2\tau}{\kappa} \partial_s \kappa + \partial_s \tau \right) |_{\partial \gamma} \end{aligned}$$

Since the boundary of  $\gamma$  is empty,

$$\partial_t \|\tau\|_1 = \int_{\gamma_t} \kappa^2 \cdot |\tau| ds > 0.$$

Therefore the  $L^1$  norm is positive and increasing and so approaches a positive (possibly infinite) limit as  $t$  goes to  $\omega$ . However,

$$\sup_{p \in \gamma_t} \tau(p) \cdot L_t \geq \|\tau\|_1(t) > 0$$

so  $\liminf_{t \rightarrow \omega} \sup_{p \in \gamma_t} \tau(p) \cdot L_t \geq \lim_{t \rightarrow \omega} \|\tau\|_1(t) = C > 0$ .

But then

$$\begin{aligned} \limsup_{t \rightarrow \omega} \sup_{p \in \gamma_t} \frac{\tau}{\kappa} &\geq \frac{C}{D} > 0. \\ &\Rightarrow \Leftarrow \end{aligned}$$

Therefore  $\gamma$  cannot develop a type I singularity and so develops a type II singularity.  $\square$

### 3. GOING FORWARD

A standard maximum principle argument shows that if a curve develops a type I singularity, if its torsion is ever everywhere positive after the last inflection point at time  $c$ , it remains positive for all time. Suppose there is a time  $d > c$  such that  $\inf_{p \in \gamma_d} \tau = 0$ . Suppose that  $\tau(p) = 0$ . Then,

$$\begin{aligned} \partial_t \tau(p) &= \partial_s^2 \tau + 2 \frac{1}{\kappa} (\partial_s \kappa) (\partial_s \tau) + \frac{2\tau}{\kappa} \left( \partial_s^2 \kappa - \frac{1}{\kappa} (\partial_s \kappa)^2 + \kappa^3 \right) \\ &= \partial_s^2 \tau + 2 \frac{1}{\kappa} (\partial_s \kappa) (\partial_s \tau) \text{ since } \tau = 0. \\ &= \partial_s^2 \tau \geq 0 \text{ since } \partial_s \tau = 0 \text{ and } p \text{ minimizes } \tau. \end{aligned}$$

Therefore, after the last inflection point, there is always at least two points for which torsion vanishes and the torsion has positive and negative sections. Furthermore, for any type I singularity, the  $L^1$  norm of torsion must go to zero at the singularity.

### REFERENCES

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OHIO STATE UNIVERSITY, COLUMBUS, OH 43210

*E-mail address:* `khan.375@osu.edu`